Correlation Functions in Classical Statistical Physics

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$$U_n(f_1, \dots, f_n) = \frac{\partial^n}{\partial t_n \dots \partial t_1} \log \left(\mu(e^{\sum_{i=1}^n t_i f_i}) \right) |_{t_1 = \dots = t_n = 0} \ .$$

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- For the sake of concreteness, consider the Ising model on $\mathbb{Z}^d$ given by the (formal) Hamiltonian:

$$H = - \sum_{\{i,j\} \subset \mathbb{Z}^d} J_{i,j} \sigma_i \sigma_j$$

with $\sigma := (\sigma_i)_{i \in \mathbb{Z}^d} \in \{\pm 1\}$, $J_{i,j} \geq 0$ and the Boltzmann distribution:

$$\mathbb{P}_\beta(\omega) \propto e^{-\beta H(\omega)}.$$

with $\beta \geq 0$ and some configuration $\omega \in \{\pm 1\}^{\mathbb{Z}^d}$.

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We also define the inverse *critical temperature* by

$$\beta_c = \inf\{\beta \geq 0 : \inf_{x \in \mathbb{Z}^d} \langle \sigma_0 \sigma_x \rangle_\beta > 0\}.$$

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- Any local local functions f and g, there exist $c_A^f, c_B^g \in \mathbb{R}$ such that

$$f = \sum_{A \subset \text{supp}(f)} c_A^f \sigma_A \qquad g = \sum_{B \subset \text{supp}(f)} c_B^g \sigma_B,$$

with $\sigma_A := \prod_{i \in A} \sigma_i$. In particular, one has

$$\langle f; g \rangle_\beta := \text{Cov}_{\mathbb{P}_\beta}[f, g] = \sum_{\substack{A \subset \text{supp}(f) \\ B \subset \text{supp}(g)}} c_A^f c_B^g \langle \sigma_A; \sigma_B \rangle_\beta.$$

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Therefore, understanding $\langle f; g \rangle_\beta$ **amounts to** understanding $\langle \sigma_A; \sigma_B \rangle_\beta$.

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Question: What can be said about the asymptotic behaviour of $\langle \sigma_0 ; \sigma_x \rangle_\beta$ as $\|x\| \to \infty$?

- In 1914 and 1916, Ornstein and Zernike developed a (heuristic) theory of correlations with quickly decaying interactions. In particular, they concluded that, at large distances away from the critical temperature, the spin-spin correlation of the Ising model satisfies

$$\langle \sigma_0; \sigma_x \rangle_\beta \sim \|x\|^{-(d-1)/2} \, e^{-\nu_\beta(x)}.$$

- **Is it possible** to establish this result **rigourously** ?

OZ sharp asymptotics when $\beta < \beta_c$

- One has the following **Ornstein–Zernike asymptotics**:

Theorem [A.–Ott–Velenik 2021]

Assume that $\beta < \beta_c$. Let $\vec{s} \in \mathbb{S}^{d-1}$. Then, as $n \to \infty$,

$$\langle \sigma_0 \, ; \sigma_{n\vec{s}} \rangle_\beta = \frac{\Psi_\beta(\vec{s})}{n^{(d-1)/2}} \, e^{-\nu_\beta(\vec{s})n} \, (1 + o(1)),$$

where the functions Ψ_β and $\nu_\beta(\vec{s})$ are positive and analytic in $\vec{s}$.

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- The above result has a long history. Some milestones are
  - *Ornstein–Zernike 1914, Zernike 1916:* first (non-rigorous) derivation
  - *Wu 1966, Wu et al 1976:* exact computation, planar model, $\beta < \beta_c$
  - *Abraham–Kunz 1977, Paes-Leme 1978:* any dimension, n.n. model, $\beta \ll 1$
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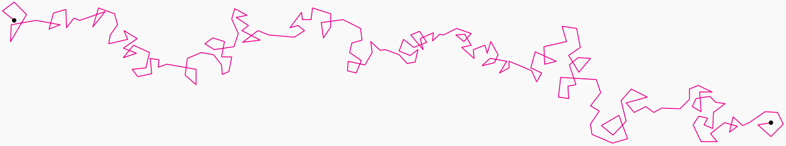
Probabilistic picture behind OZ asymptotics

- In order to study the subcritical Ising model, one can different graphical representations, for instance the **high temperature expansion**

$$\langle \sigma_0; \sigma_x \rangle_\beta = \sum_{\gamma : 0 \to x} q_\beta(\gamma).$$

- The techniques developped during last two decades allow to **couple "structurally 1D objects"** with good mixing properties **to directed random walks**.



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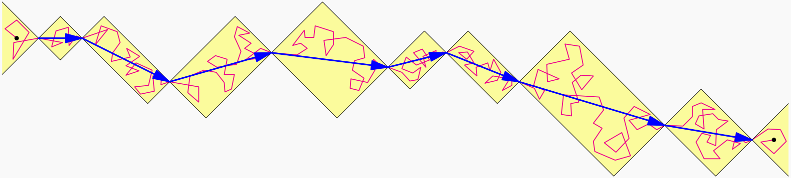

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- Using this coupling, we can in many cases **reduce difficult questions arising in the Ising model to much simpler (and more classical) ones about random walks**.

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Odd-odd correlations

$|A|$, $|B|$ both odd

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Odd-odd correlations	Even-even correlations
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- In the Odd–Odd case, the OZ asymptotics still hold.

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- However, concerning the prefactor, two conflicting predictions were put forward:

Polyakov 1969		Camp, Fisher 1971	
n^{-2}	$d = 2$		
$(n \log n)^{-2}$	$d = 3$	n^{-d}	for all $d \geq 2$
$n^{-(d-1)}$	$d \geq 4$		

(Note that these predictions only coincide when $d = 2$, where they both agree with the exact computation obtained in *Stephenson 1966* and *Hecht 1967*.)

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- It turns out that Polyakov was right. This was first shown in
  - *Bricmont–Fröhlich 1985:* $|A| = |B| = 2$ $\quad \beta \ll 1$ $\quad d \geq 4$
  - *Minlos–Zhizhina 1988, 1996:* $|A|, |B|$ even $\quad \beta \ll 1$ $\quad d \geq 2$

- The best nonperturbative result to date is the following:

$$\text{Let} \quad \tau(n) = \begin{cases} n^2 & \text{when } d = 2, \\ (n \log n)^2 & \text{when } d = 3, \\ n^{d-1} & \text{when } d \geq 4. \end{cases}$$

Theorem [Ott–Velenik 2019]

Let $d \geq 2$ and $\beta < \beta_c$. Let $A, B \Subset \mathbb{Z}^d$ with $|A|$ and $|B|$ even and let $\vec{s} \in \mathbb{S}^{d-1}$. Then, there exist constants $0 < C_- \leq C_+ < \infty$ (depending on $A, B, \vec{s}, \beta$) such that, for all n large enough,

$$\frac{C_-}{\tau(n)} e^{-2\nu_\beta(\vec{s})n} \leq \langle \sigma_A \,;\, \sigma_{B+n\vec{s}} \rangle_\beta \leq \frac{C_+}{\tau(n)} e^{-2\nu_\beta(\vec{s})n}.$$

New directions and open problems

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- The OZ decay was proved in the ground state of the quantum Ising model in a strong transverse magnetic field (Kennedy 1991). Could it be done more generally with the modern OZ theory ?

Thank you and buen apetito!